Spectrum Sensing in Cognitive Radios using Distributed Sequential Detection

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Outline

- Introduction
- Parametric Distributed Sequential Tests
- Nonparametric Distributed Sequential Tests
- Multihypothesis scenario
- Conclusions

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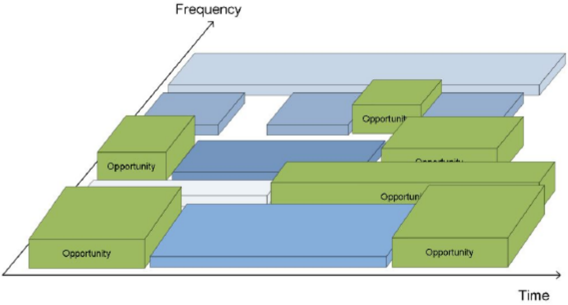

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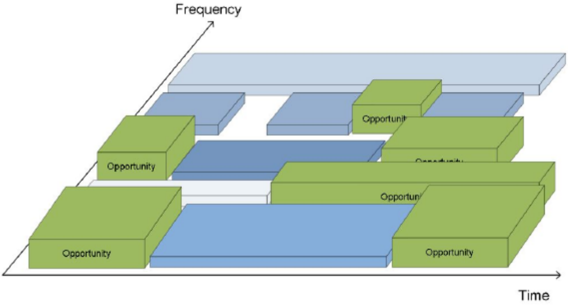

- Aided by evolving SDR technology

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- Mitigates multipath fading, shadowing and hidden node problem.
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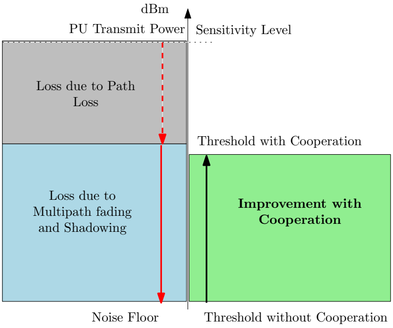

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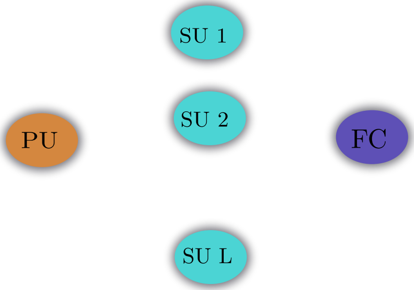

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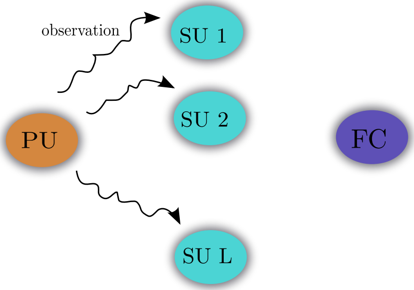

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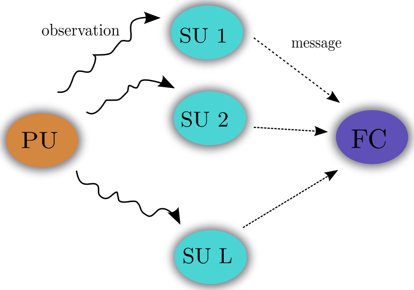

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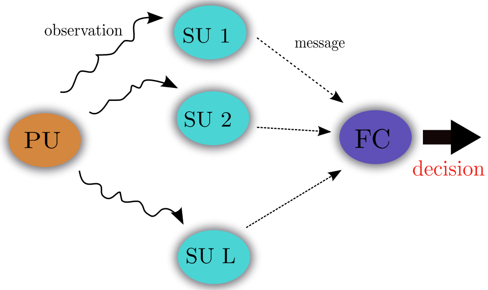

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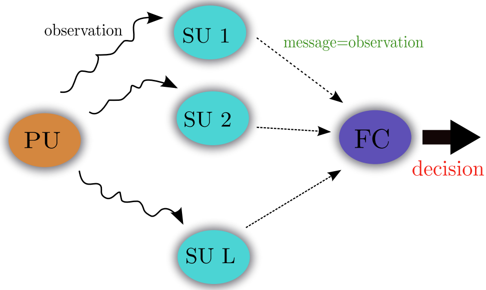

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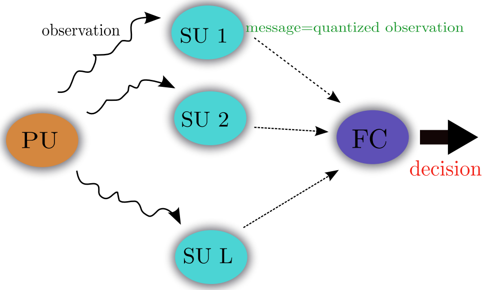

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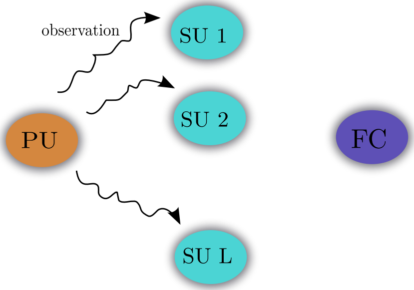

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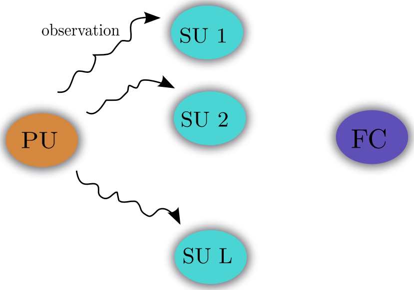

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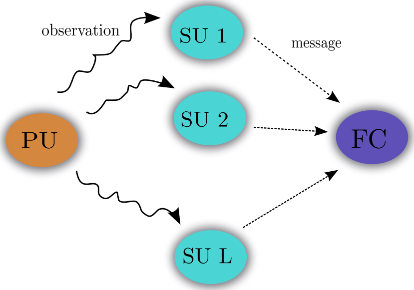

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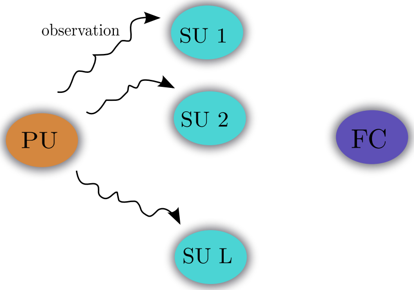

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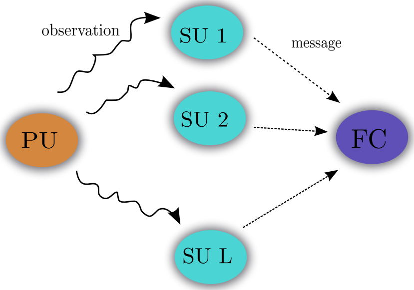

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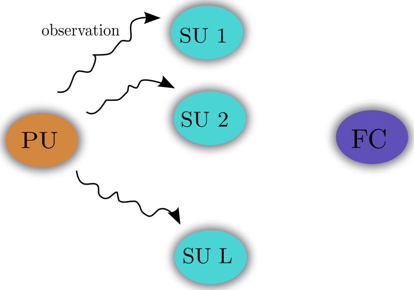

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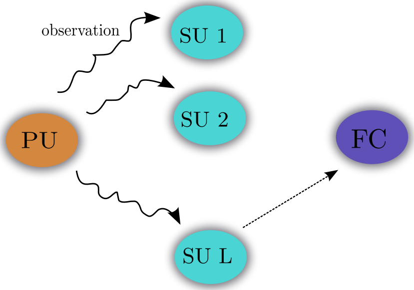

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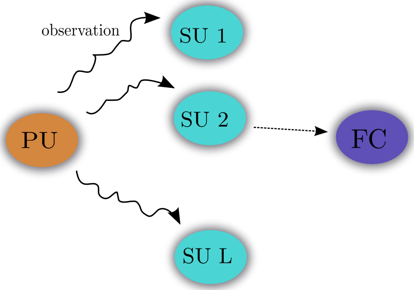

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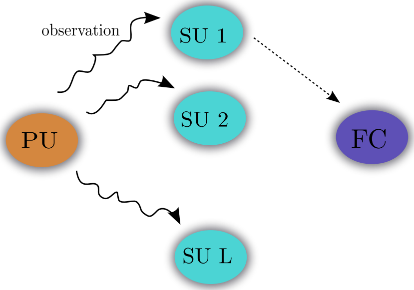

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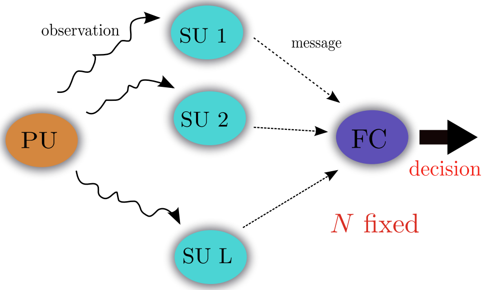

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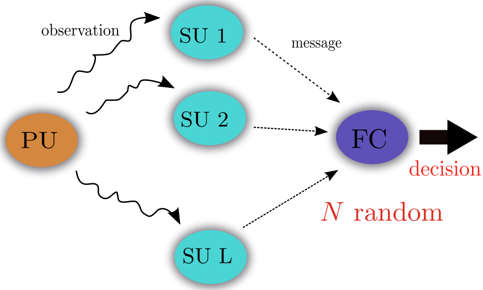

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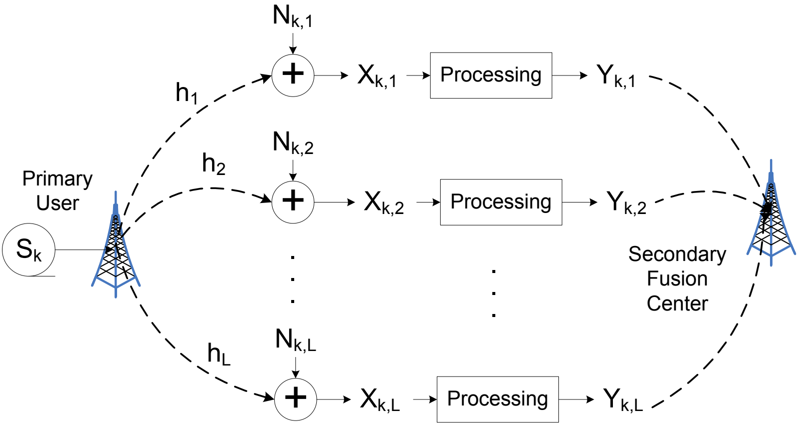
- Detect the spectrum and identify the primary user
  (Decentralized Multihypothesis Sequential Testing)

Related Work

- Sequential tests: Wald'47, Irle'84
- Decentralized Sequential tests: Mei'08, Fellouris et al.'12, Yilmaz et al.'12
- Cooperative Spectrum Sensing and Censored detection: Akyildiz et al.'11
- Universal hypothesis tests: Levitan et al.'02, Jacob et al.'08, Unnikrishnan et al.'11
- Multihypothesis Sequential tests: Draglia et al.'99, Tartakovsky'00

Model



$N_{k,1}$
h_1
$X_{k,1}$
Processing
$Y_{k,1}$
$N_{k,2}$
h_2
$X_{k,2}$
Processing
$Y_{k,2}$
Primary User
S_k
Secondary Fusion Center
h_L
$N_{k,L}$
$X_{k,L}$
Processing
$Y_{k,L}$

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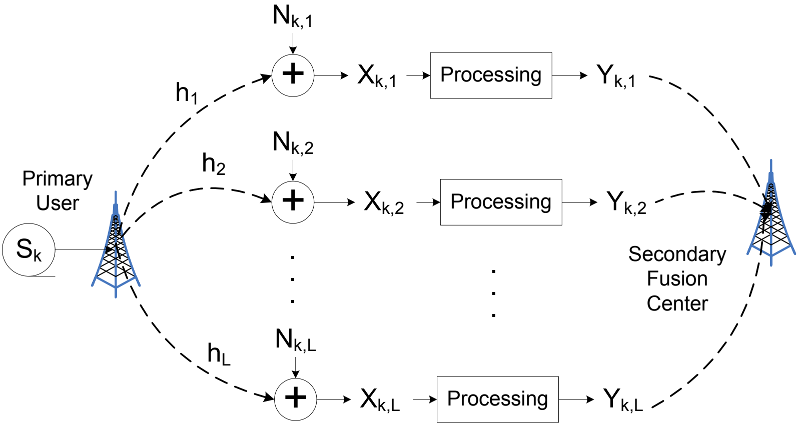

General Problem: N = Stopping time, δ_N = decision rule

$$\min_{N,\delta_N} E[N|H_0] \text{ and } \min_{N,\delta_N} E[N|H_1],$$

subject to $P_{FA} \le \alpha_1$ and $P_{MD} \le \alpha_2$

Parametric Distributed Sequential Tests

- At l $^{\text{th}}$ CR,

$$\begin{aligned} H_0 : \quad & X_{k,l} \sim f_{0,l} \\ H_1 : \quad & X_{k,l} \sim f_{1,l} \end{aligned}$$

- $f_{0,l}$ and $f_{1,l}$ are fully known
- $f_{0,l}$ and $f_{1,l}$ are not fully known: parametric family.

DualSPRT Algorithm

1. At Node, l, Sequential Probability Ratio Test (SPRT),

$$W_{k,l} = W_{k-1,l} + \log \frac{f_{1,l}(X_{k,l})}{f_{0,l}(X_{k,l})},\ W_{0,l} = 0.$$

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$$Y_{k,l} = b_1 1_{\{W_{k,l} \geq \gamma_1\}} + b_0 1_{\{W_{k,l} \leq -\gamma_0\}}$$

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$$Y_k = \sum_{l=1}^{L} Y_{k,l} + Z_k$$

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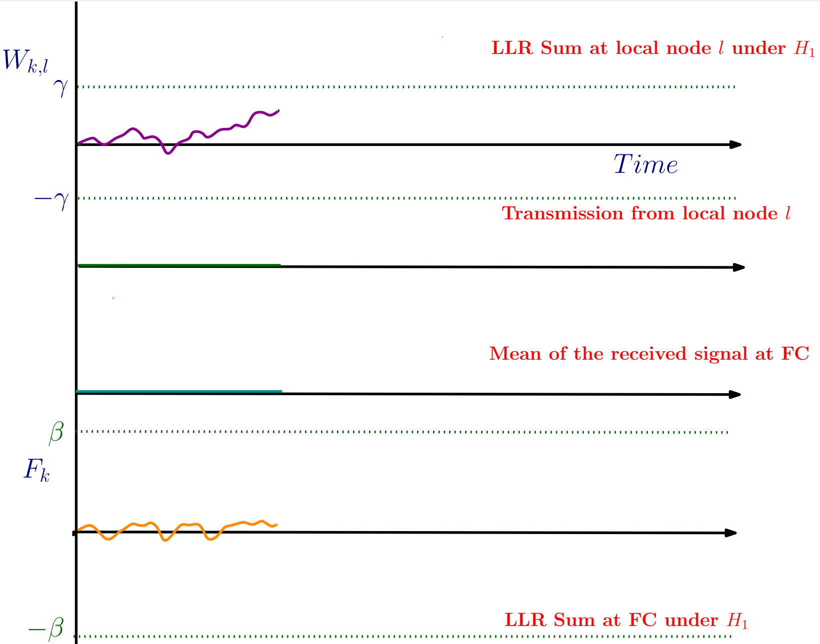
4. FC decides, at $N = \inf\{k : F_k \notin (-\beta_0, \beta_1)\}$, H_1 if $F_N \geq \beta_1$, H_0 if $F_N \geq -\beta_0$.

Performance Analysis of DualSPRT- E_{DD}

Sample path argument



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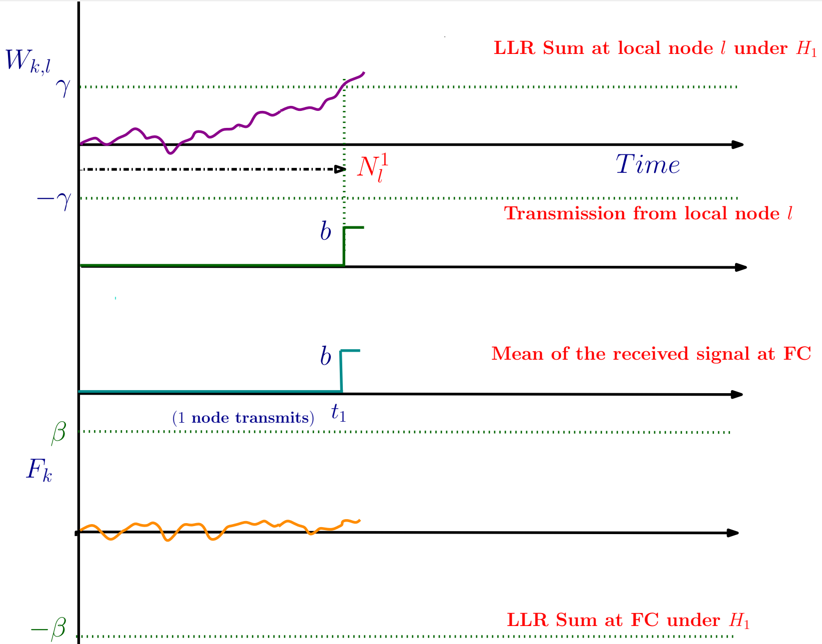

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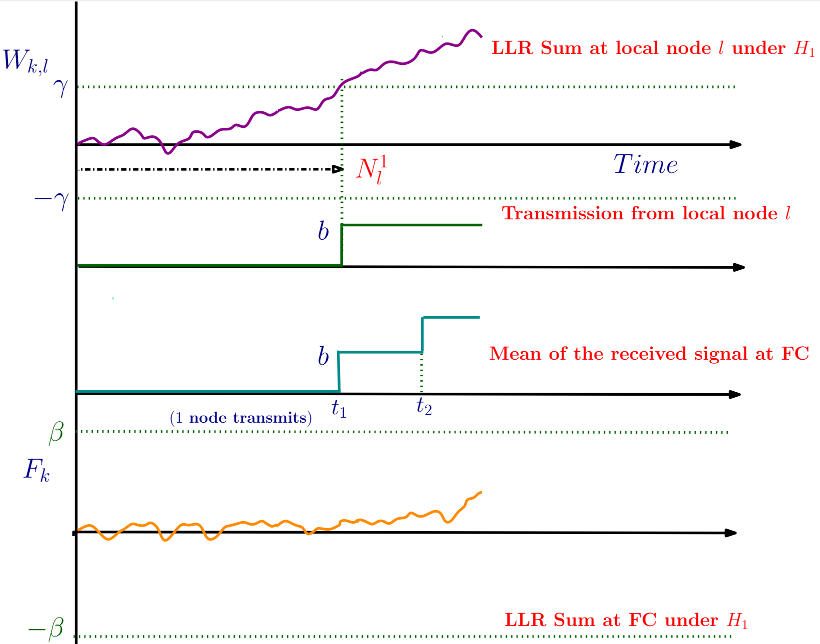
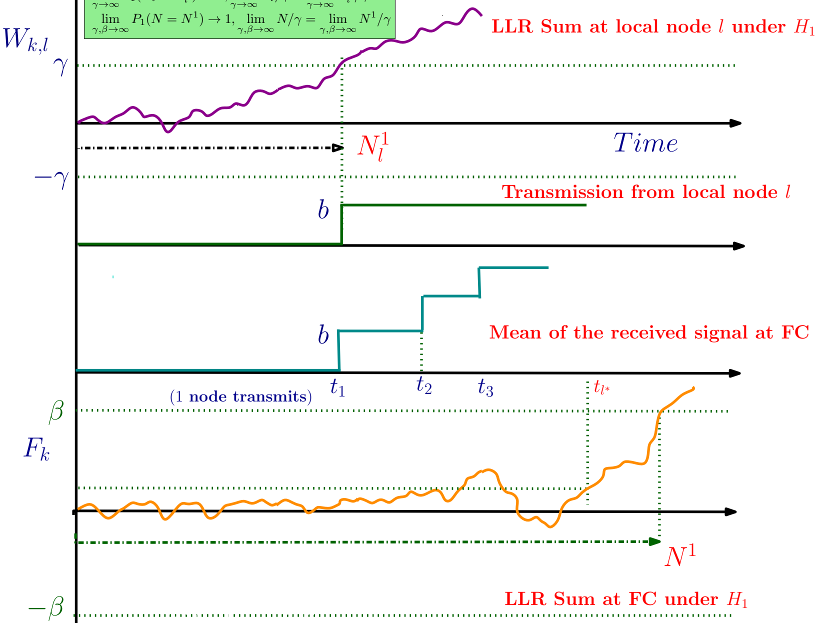


Performance Analysis of DualSPRT- E_{DD}

Sample path argument

$$\lim_{\gamma\to\infty} P_1(N_l = N_l^1) \to 1, \lim_{\gamma\to\infty} N_l/\gamma = \lim_{\gamma\to\infty} N_l^1/\gamma$$
$$\lim_{\gamma,\beta\to\infty} P_1(N = N^1) \to 1, \lim_{\gamma,\beta\to\infty} N/\gamma = \lim_{\gamma,\beta\to\infty} N^1/\gamma$$



Performance Analysis of DualSPRT- E_{DD} (Contd.)

- At each node l, time to cross the threshold

$$N_l^1 \sim \mathcal{N}(\frac{\gamma}{\delta_l}, \frac{\rho_l^2 \gamma}{\delta_l^3}),\ \delta_l = E_1[LLR_l] \text{ and } \rho_l^2 = Var_1[LLR_l]$$

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- $l^* = min\{j : \delta_{FC}^j > 0 \text{ and } \frac{\beta - \bar{F}_j}{\delta_{FC}^j} < E[t_{j+1}] - E[t_j]\}$
- $\bar{F}_j = E[F_{t_j - 1}]$, $\bar{F}_j = \bar{F}_{j-1} + \delta_{FC}^j \left(E[t_j] - E[t_{j-1}]\right)$, $\bar{F}_0 = 0$.

$$E_{DD}\left(E_1[N]\right) \approx E_1[N^1] \approx E[t_{l^*}] + \frac{\beta - \bar{F}_{l^*}}{\delta_{FC}^{l^*}}$$

Performance Analysis of DualSPRT- P_{MD}

$$P_{MD} = P_1(N^0 < N^1) \geq P_1(N^0 < t_1, N^1 > t_1)$$

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\text{Also } P_1(N^0 < N^1) &\leq P_1(N^0 < \infty) \\
&= P_1(N^0 < t_1) + P_1(t_1 \leq N^0 < t_2) \\
&\quad + P_1(t_2 \leq N^0 < t_3) + \dots
\end{aligned}
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$$P_{MD} \approx P_1(N^0 < t_1)$$

Performance Analysis of DualSPRT- P_{MD}

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\end{aligned}
$$

$$
\begin{aligned}
P_{MD} &\approx {\color{red} P_1(N^0 < t_1)} \\
&= \sum_{k=1}^{\infty} P\left[\{F_k < -\beta\} \cap_{n=1}^{k-1} \{F_n > -\theta\} \middle| t_1 > k\right] P[t_1 > k]
\end{aligned}
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- 5 Secondary nodes
- channel gains (0, -1.5, -2.5, -4 and -6 dB)

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$P_{MD} Sim.$	$P_{MD} Anal.$	$E_{DD} Sim.$	$E_{DD} Anal.$
$18.78e-4$	$19.85e-4$	44.319	43.290
$26.68e-4$	$27.51e-4$	36.028	34.634
$36.30e-4$	$35.16e-4$	27.770	25.977

Asymptotic Properties of DualSPRT

$$\mathcal{R}_c(\delta) = \pi[c\, E_0(N) + W_0 P_0\{reject\ \ H_0\}] + (1-\pi)[c\, E_1(N) + W_1 P_1\{reject H_1\}]$$

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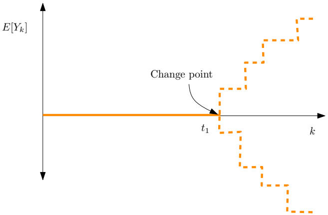

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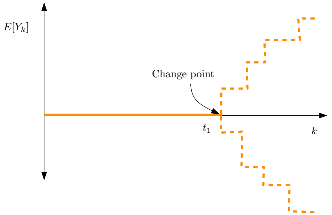

- Reducing the false alarms caused by FC MAC noise before t_1.

- SPRT-CSPRT: (inspired from CUSUM) FC algorithm,

$$F_k^1 = \left(F_{k-1}^1 + \log \frac{g_{\mu_1}(Y_k)}{g_z(Y_k)}\right)^+, \; F_k^0 = \left(F_{k-1}^0 + \log \frac{g_z(Y_k)}{g_{-\mu_0}(Y_k)}\right)^-$$

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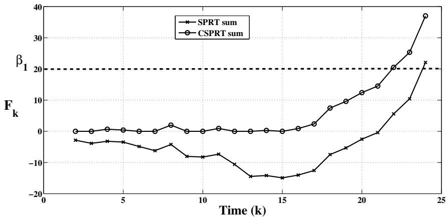
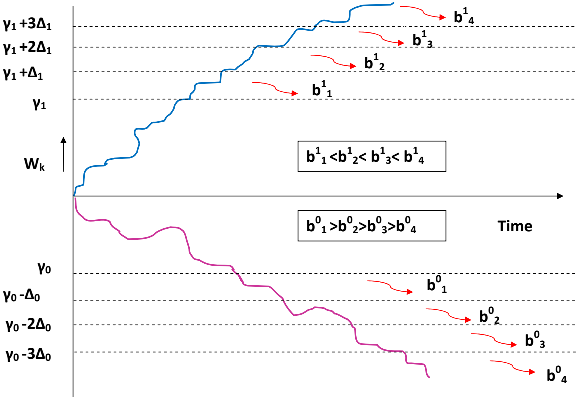
- Sample path argument



Modification of quantisation at SU's



Performance Analysis of SPRT-CSPRT: P_{MD}

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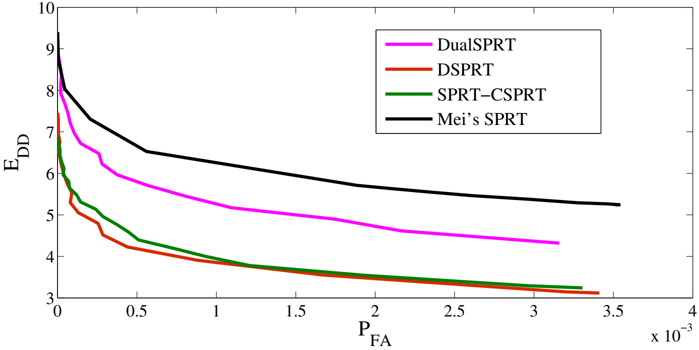
$P_{MD} Sim.$	$P_{MD} Anal.$	$E_{DD} Sim.$	$E_{DD} Anal.$
0.00675	0.00613	26.8036	24.9853
0.0072	0.0065	33.1585	31.7624
0.01675	0.01624	30.0817	29.1322

Comparison with asymptotically optimal tests



Mei's SPRT: Mei, Trans. Inf. Theory 2008.

DSPRT: Fellouris and Moustakides, Trans. Inf. Theory 2011.

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- SNR at local CR nodes Unknown

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$$\begin{aligned} N &= \inf \{ n : W_{n,l} \geq g(cn) \} \, , \\ g(t) &\approx \log(1/t) \text{ as } t \to 0 \end{aligned}$$

- Decide upon H_0 or H_1 as $\hat{\theta}_N \leq \theta^*$ *or* $\hat{\theta}_N \geq \theta^*$ where θ^* solves $D(f_{\theta^*}||f_{\theta_0}) = D(f_{\theta^*}||f_{\theta_1})$
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- $D(f_\theta||f_\lambda) = E_\theta[\log[f_\theta(X)/f_\lambda(X)]$
- Similar to GLR test in Neyman-Pearson approach in FSS tests
- Diminishing uncertainty of $\hat{\theta}_n$ as an estimate of θ with n is incorporated into the varying stopping boundary $g(cn)$
- Asymptotically optimal over a broad range of θ as $c \to 0$

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hyp	E_{DD}	$P_E = 0.1$	$P_E = 0.05$	$P_E = 0.01$
H1	DualSPRT	2.06	3.177	5.264
H1	GLRSPRT	1.425	2.522	4.857
H0	DualSPRT	1.921	3.074	5.184
H0	GLRSPRT	2.745	3.852	6.115

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$$H_0 : f_{0,l} \sim \mathcal{N}(0, \sigma^2); H_1 : f_{1,l} \sim \mathcal{N}(\theta, \sigma^2)$$

where θ is exponential distributed r.v which reflects the unknown channel gain.

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E_{DD}	$P_{MD} = 0.1$	$P_{MD} = 0.08$	$P_{MD} = 0.06$
DualSPRT	1.74	1.854	2.417
GLRSPRT	1.62	3.065	5.42

Table: slow-fading between primary and secondary user under H1

Comparison between SPRT-CSPRT and GLR-CSPRT

- 5 Secondary nodes
- Unknown SNR: channel gains (0, -1.5, -2.5, -4 and -6 dB)
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Hyp	E_{DD}	$P_E = 0.1$	$P_E = 0.05$	$P_E = 0.01$
H_1	SPRT-CSPRT	1.615	2.480	4.28
H_1	GLR-CSPRT	1.138	2.221	4.533
H_0	SPRT-CSPRT	1.533	2.334	4.225
H_0	GLR-CSPRT	2.424	3.734	5.72

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Hyp	E_{DD}	$P_E = 0.1$	$P_E = 0.07$	$P_E = 0.04$
H_1	SPRT-CSPRT	1.03	1.53	2.347
H_1	GLR-CSPRT	0.94	1.004	4.225
H_0	SPRT-CSPRT	1.528	1.741	2.415
H_0	GLR-CSPRT	2.615	3.192	4.237

Universal Sequential Hypothesis Testing

- Nonparametric or universal setup:
  - Noise statistics under no PU transmission is fully known
  - Transmit power, channel gains, modulation schemes etc. of PU transmissions is not available (SNR uncertainty).
- i.i.d. observations $X_i, i = 1, 2, \ldots$
- $H_0 : X_i \sim P_0($ p.d.f. $f_0)$ known ;
  $H_1 : X_i \sim P_1($ p.d.f. $f_1)$ unknown

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- P_0 is known; P_1 is not known. Replace W_n by

$$\widehat{W}_n = -L_n(X_1^n) - \log P_0(X_1^n) - n\frac{\lambda}{2},\ \lambda > 0.$$

$L_n(X_1^n)$=Codelength of a universal lossless source code for X_1^n

- By Shanon-Macmillan Thorem, for any stationary ergodic source $\lim_{n\to\infty} n^{-1} \log P(X_1^n) = -\bar{H}(X)$ a.s. ($\bar{H}(X)$ is the entropy rate).

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  Thus consider P_1 belonging to the class

$$\mathcal{C} = \{P_1 : D(P_1, P_0) \geq \lambda\}.$$

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(5) Under (A), if $E_1[(\log P_1(X_1))^{p+1}] < \infty$ & $E_1[(\log P_0(X_1))^{p+1}] < \infty$ for some $p \geq 1$, then for all $0 < q \leq p$,

$$\frac{E_0[N^q]}{|\log \gamma_0|^q} \xrightarrow[\gamma_0,\gamma_1 \to \infty]{} \left(\frac{2}{\lambda}\right)^q, \quad \frac{E_1[N^q]}{|\log \gamma_1|^q} \xrightarrow[\gamma_0,\gamma_1 \to \infty]{} \left(\frac{1}{D(P_1||P_0) - \frac{\lambda}{2}}\right)^q$$

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Why Uniform Quantization?

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- Range of the quantization: f_1's tail probabilities less than a small specific value at a fixed boundary

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  C depends on size of the quantized alphabet.
- LZSLRT performance is close to the nearly optimal parametric GLR sequential tests and is better for some class of distributions (e.g. Pareto)

KTSLRT algorithm

- $L_n(X_{1:n}^{\Delta})$ from combined use of Krichevsky-Trofimov estimator and the Arithmetic Encoder

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  - Estimate source distribution via K-T estimator for a finite alphabet source as,

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- Universal codes defined via the K-T estimator and AE are nearly optimal

Performance Comparison LZLRT

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Table: $f_0 \sim \mathcal{N}(0,5)$ and $f_0 \sim \mathcal{N}(3,5)$

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Table: $f_0 \sim \mathcal{P}(10,2)$ and $f_0 \sim \mathcal{P}(3,2)$

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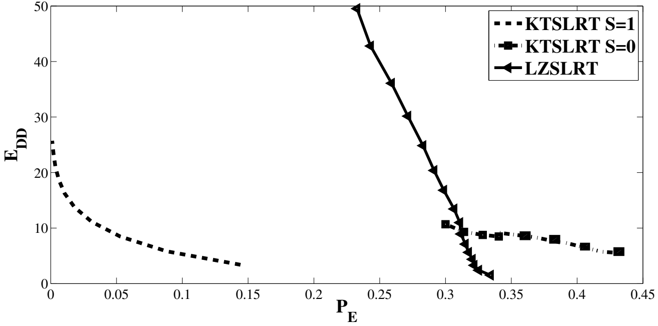
$f_1 \sim \mathcal{N}(0,5)$ and $f_0 \sim \mathcal{N}(0,1)$, 8 bit uniform quantizer.



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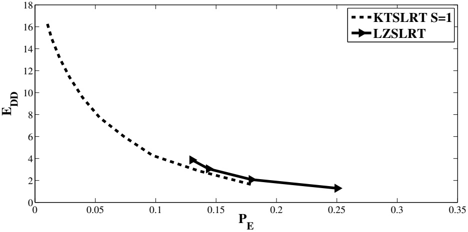

Performance Comparison

Lognormal Distribution and Pareto Distribution examples

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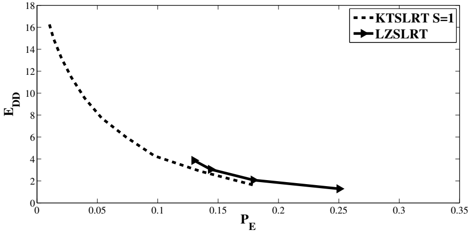
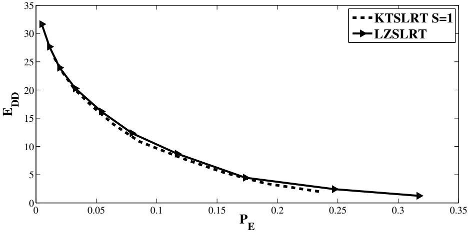


$f_1 \sim \mathcal{P}(3,2)$ and $f_0 \sim \mathcal{P}(10,2)$, support set $(2,10)$



Performance Comparison

Gaussian case ($f_1 \sim \mathcal{N}(0,5)$ and $f_0 \sim \mathcal{N}(0,1)$) with different estimators

1-NN differential entropy estimator is, $\gamma =$ Euler-Mascheroni constant

$$\hat{h}_n = \frac{1}{n}\sum_{i=1}^{n} \log \rho(i) + \log(n-1) + \gamma + 1,\ \rho(i) \triangleq \min_{j:1 \le j \le n, j \ne i} ||X_i - X_j||$$

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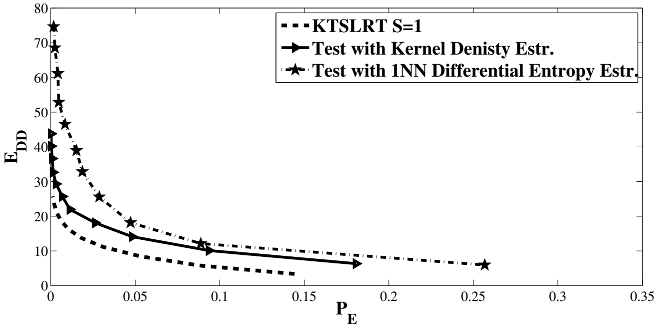

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Comparison with Hoeffding Test: $P_0 \sim B(8, 0.2)$, $P_1 \sim B(8, 0.5)$

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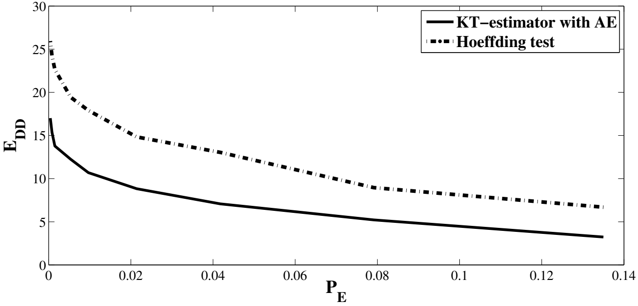

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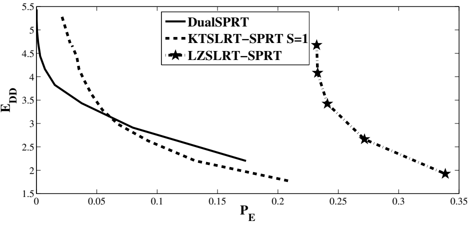
- noisy MAC between SUs and FC
- FC SPRT: binary hypothesis testing of g_{μ_1} vs $g_{-\mu_0}$

$b_1 = 1$, $b_0 = -1$, $I = 2$, $L = 5$ and $Z_k \sim \mathcal{N}(0,1)$

Decentralized case
Performance Comparison

$b_1 = 1,\ b_0 = -1,\ I = 2,\ L = 5$ and $Z_k \sim \mathcal{N}(0,1)$



$f_{0,l} \sim \mathcal{N}(0,1)$ and $f_{1,l} \sim \mathcal{N}(0,5)$, for $1 \leq l \leq L$.

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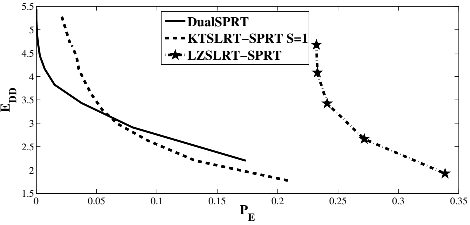

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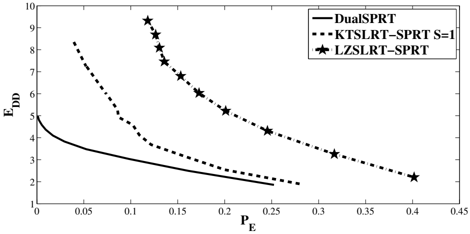

$f_{0,l} \sim \mathcal{P}(10, 2)$ and $f_{1,l} \sim \mathcal{P}(3, 2)$, for $1 \leq l \leq L$.

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Performance Comparison

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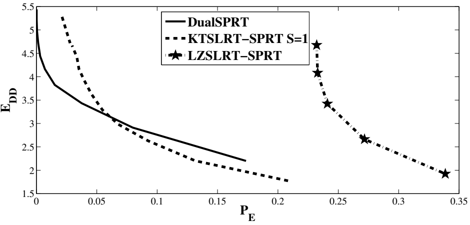

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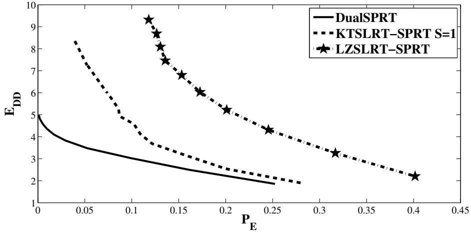

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Analysis using Perturbed Random Walk theory

$$\widehat{W_n} = S_n + \xi_n, \xi_n/n \to 0 \text{ a.s.}$$

Multihypothesis Decentralized Sequential Testing

(distributions completely known)

Multihypothesis Decentralized Sequential Testing: Algorithm-1 (DMSLRT-1)

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- Instead of physical layer fusion, TDMA used.
- FC uses the same test with hypotheses
  $G_m : Y_k \sim f_{FC}^m = \mathcal{N}(b_m, \sigma^2)$

Numerical results

$H_m : X_{k,l} \sim \mathcal{N}(m, 1),\ m{=}0,\ldots,\ 4,\ \text{No of local nodes}{=}5$

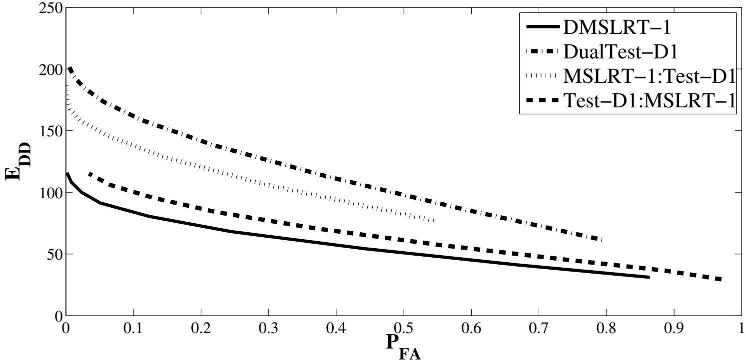

Figure: Comparison among different Multihypothesis schemes

Algorithm-2 (DMSLRT-2)

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- Expected drift of $\widehat{F}_n^{i,0} > 0$ only when $E[Y_n] > b_i/2$
- Positive b_i's make $E[F_n^{i,j}]$ negligible before first transmission

Numerical results

SNRs (-10 dB, -6 dB, 0 dB and 6 dB), PU with SNR -10 dB uses the channel.

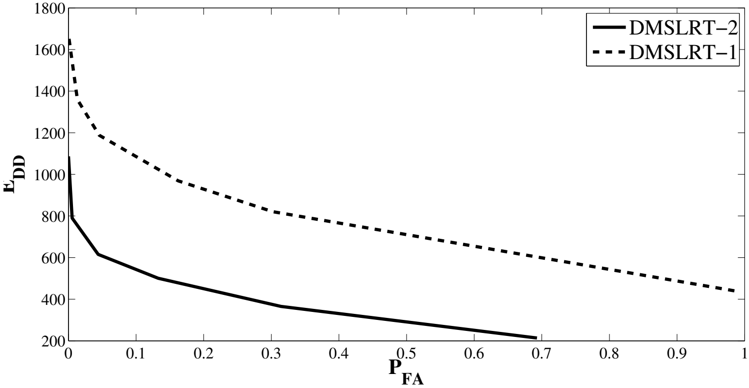

Figure: Comparison between MDSLRT-1 and MDSLRT-2

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- E_{DD} **Analysis at SU**: Dominant event-Reflected random walk with minimum positive expected drift

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- E_{DD} **Analysis of DMSLRT-1**: $E_{DD} \approx E_i[\max_l N_l] + E_i[N_{FC}]$

Comparisons of Analysis with simulations

Threshold (A)	E_{DD} Simln.	E_{DD} Anal
100	157.54	141.61
120	186.98	169.34
140	216.31	197.07

Table: At SU: Comparison of E_{DD} obtained via simulation and analysis.

Comparisons of Analysis with simulations

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Table: At SU: Comparison of E_{DD} obtained via simulation and analysis.

Threshold (A)	P^{l}_{FA} Simln.	P^{l}_{FA} Anal
8	0.0138	0.0296
10	0.0043	0.0093
20	$5.00E-5$	$1.81E-5$

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A	B	E_{DD} Simln.	E_{DD} Anal
10	80	116.79	133.63
10	90	144.04	147.49
10	100	163.54	161.36

Table: DMSLRT-1: Comparison of E_{DD} obtained via simulation and analysis.

Conclusions

- Cooperating Spectrum Sensing algorithms in sequential detection framework

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- Universal sequential tests: discrete alphabet source (asymptotic properties derived), continuous alphabet source, decentralized scenario
- Decentralized multihypothesis Sequential tests: performance analysis, numerical comparisons

Thank You

Publications based on this Thesis

1. J. K. Sreedharan and V. Sharma, "Spectrum sensing using distributed sequential detection via noisy MAC", submitted to journal.
2. J. K. Sreedharan and V. Sharma. "Nonparametric distributed sequential detection via universal source coding", in *Information Theory and Applications Workshop (ITA)*, California, USA, Feb 2013.
3. J. K. Sreedharan and V. Sharma, "Spectrum Sensing via Universal Source Coding", in *Proc. IEEE Global Communications Conference (GLOBECOM)*, California, USA, Dec 2012.
4. K. S. Jithin and V. Sharma, "Novel algorithms for distributed sequential hypothesis testing", in *Proc. 49th Annual Allerton Conference on Communication, Control and Computing*, Illinois, USA, Sep 2011 (invited paper).
5. J. K. Sreedharan and V. Sharma, "A novel algorithm for cooperative distributed sequential spectrum sensing in Cognitive Radio", in *Proc. IEEE Wireless Communications and Networking Conference (WCNC)*, Cancun, Mexico, Mar 2011.
6. K. S. Jithin, V. Sharma, and R. Gopalarathnam, "Cooperative distributed sequential spectrum sensing", in *Proc. IEEE National Conference on Communication (NCC)*, Bangalore, India, Jan 2011.

Lemmas to support DualSPRT analysis

Lemma 1

For $i = 0, 1$, $P_i(N_l = N_l^i) \to 1$ as $\gamma \to \infty$ and $P_i(N = N^i) \to 1$ as $\gamma \to \infty$ and $\beta \to \infty$.

Lemma 2

Under H_i, $i = 0, 1$ and $j \neq i$,

(a) $|N_l - N_l^i| \to 0$ a.s. as $\gamma \to \infty$ and $\lim_{\gamma \to \infty} \frac{N_l}{\gamma} = \lim_{\gamma \to \infty} \frac{N_l^i}{\gamma} = \frac{1}{D(f_{i,l} || f_{j,l})}$ a.s. and in L^1.

(b) $|N - N^i| \to 0$ a.s. and $\lim \frac{N}{\beta} = \lim \frac{N^i}{\beta}$ a.s. and in L^1, as $\gamma \to \infty$ and $\beta \to \infty$.

Lemma 3

Let t_k be the time when k local nodes have made the decision. Under H_i, $i = 0, 1$, as $\gamma \to \infty$,
$P_i(\text{Decision at time } t_k \text{ is } H_i \text{ and } t_k \text{ is the } k^{th} \text{ order statistics of } N_1^i, N_2^i, \ldots, N_L^i) \to 1$.

CR block diagram

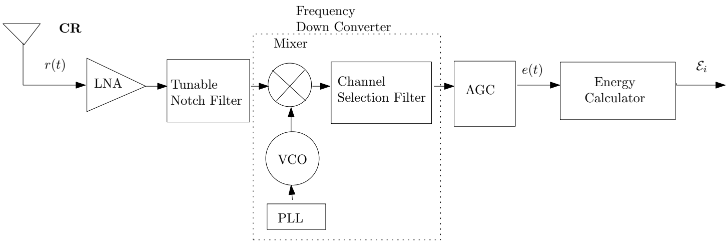

Figure: Block diagram of the receiver implementation at a CR

$P_{H_1}(FA\ before\ t_1)$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} P\Big[\{F_k < -\theta\} \cap_{n=1}^{k-1} \{F_n > -\theta\}\Big| t_1 > k\Big] P[t_1 > k] \\
&= \sum_{k=1}^{\infty} \Big(P[F_k < -\theta| \cap_{n=1}^{k-1} \{F_n > -\theta\}]\ P[\cap_{n=1}^{k-1}\{F_n > -\theta\}]\Big) \\
&\quad P[t_1 > k] \\
&\overset{(A)}{=} \sum_{k=1}^{\infty} \Big(P[F_k < -\theta|F_{k-1} > -\theta]\ P[\inf_{1\le n\le k-1} F_n > -\theta]\Big) \\
&\quad P[t_1 > k] \\
&\overset{(B)}{\geq} \sum_{k=1}^{\infty} \Big(\int_{c=0}^{2\theta} P[S_k < -c] f_{F_{k-1}}\{-\theta + c\} dc\Big) \\
&\quad \Big(1 - 2P[F_{k-1} < -\theta]\Big)\Big(\prod_{l=1}^{L}(1 - \Phi_{\tau_{\gamma,l}}(k)\Big)
\end{aligned}
$$

(A) is because of the Markov property of the random walk. (B) is due to the inequality, $P[\sup_{k\le n} F_k \geq \theta] \leq 2P[F_n \geq \theta]$ for the Gaussian R.W

Gaussian Mean change approximation of Energy Detector

- $X_{k,l}$ are a summation of energy of N samples received by the l^{th} Cognitive Radio
- For large N, the pre and post change distributions of $X_{k,l}$ can be approximated by Gaussian distributions: $f_{0,l} \sim \mathcal{N}(\sigma^2, 2\sigma^4/N)$ and $f_{1,l} \sim \mathcal{N}(P_l + \sigma^2, 2(P_l + \sigma^2)^2/N)$ where P_l is the received power at the l^{th} CR node and noise $Z_{k,l} \sim \mathcal{N}(0, \sigma^2)$.
- Under low SNR conditions $(P_l + \sigma^2)^2 \approx \sigma^4$ and hence $X_{k,l}$ are Gaussian distributed with mean change under H_0 and H_1
- Take $X_{k,l} - \sigma^2$ as the data for l^{th} node SPRT.

Gaussian Mean change under log normal modelling

- $X_{k,l}$ is the received power in decibels.
- pre change distribution of $X_{k,l}$ is $\mathcal{N}(\mu_0, {\sigma_l}^2)$ and post change distribution $\mathcal{N}(\mu_0 + P_l, {\sigma_l}^2)$
- Secondary nodes are using Energy detector
- Under H_0 the uncertainty in noise and interference power is assumed to be log normally distributed. i.e log of the received power is Gaussian. μ_0 is mean noise power. ${\sigma_l}^2$ is the uncertainty in noise power.
- Under H_1, P_l- mean increase in received power due to the presence of primary. $P_l = 10 \log_{10}(1 + SNR)$ dB.
- Log normal distribution is valid under H_1 as its used for modelling shadowing.

Nonlinear renewal theory argument for MSLRT-1 analysis

Under mild conditions the limiting distribution of the excess of a random walk over a fixed threshold does not change by the addition of a slowly changing nonlinear term.

$$E_i[N_l] \approx \frac{A + \mathcal{X}_l^{i,\widehat{j}} + \mathcal{B}_l^{i,\widehat{j}}}{D(f_l^i || f_l^{\widehat{j}})},$$

where $\mathcal{X}_l^{i,\widehat{j}} = \frac{E_i[(R_{1,l}^{i,\widehat{j}})^2]}{2E_i[(R_{1,l}^{i,\widehat{j}})^2]} - \sum_{n=1}^{\infty} n^{-1} E_i S_{n,l}^{-\,i,\widehat{j}}$ and $\mathcal{B}_l^{i,\widehat{j}} = -\sum_{n=1}^{\infty} n^{-1} E_i S_{n,l}^{-\,i,\widehat{j}}$

$$R_{k,l}^{i,\widehat{j}} = \log(f_l^i(X_{k,l}/f_l^{\widehat{j}}(X_{k,l}) \text{ and } S_{n,l}^{-\,i,\widehat{j}} = -\min(0, \sum_{k=1}^{n} R_{k,l}^{i,\widehat{j}}).$$